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AN AXISYMMETRIC ANALOGUE OF THE KELDYSH-SEDOV PROBLEM*

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An axisymmetric harmonic problem is considered for a half-space on whose boundary there is any number of cirular concentric lines separating boundary conditions of the 1st and 2nd kinds. A method is given for constructing an effective solution of the problem with different relations between the geometric parameters. The problem of the joint action of ring and circular dies on an elastic half-space is taken as an example. The problem was solved earlier /1, 2/ by methods which are effective when the lines dividing the boundary conditions are remote from one another.

1. Formulation of the problem. Let s be one of the sets

$$\Omega_i^j = \bigcup_{k=i}^{j} (a_k, b_k),\ \Sigma_i^j = \bigcup_{k=i}^{j} (b_k, a_{k+1})\ (0 \leqslant a_k < b_k < a_{k+1})$$

and let $\bar{s}$ be the corresponding closed set. Let $c[\bar{s}]$ be the space of continuous functions which are given in $\bar{s}$ and decrease at infinity (if $\bar{s}$ is an unbounded set) like r^{-h} $(h > 0, r \in \bar{s})$, and let $c^*[\bar{s}]$ be the subspace of $c[\bar{s}]$ in which the functions decrease at infinity at least as fast as r^{-1-h}.

We wish to find the function $u(r, z)$ which is harmonic in the domain $r \geqslant 0$, $z > 0$ (r, z are cylindrical coordinates) and which satisfies in the $z = 0$ plane the mixed boundary conditions

$$u|_{z=0} = f(r),\ r \in \Omega_1^n,\ \partial u/\partial z|_{z=0} = g(r),\ r \in \Sigma_0^n \tag{1.1}$$
$$b_0 = 0,\ a_{n+1} = \infty,\ n < \infty,\ f \in c[\bar{\Omega}_1^n],\ g \in c^*[\bar{\Sigma}_0^n]$$

and the conditions

$$\lim_{r \to a_k+0} \sqrt{r - a_k}\, \frac{\partial u}{\partial z}\Big|_{z=0} < \infty, \quad \lim_{r \to b_k-0} \sqrt{b_k - r}\, \frac{\partial u}{\partial z}\Big|_{z=0} < \infty \tag{1.2}$$

and is decreasing at infinity. Conditions (1.2) ensure that the solution of the problem /3/ is unique and henceforth will be omitted for brevity.

We introduce into the space $c[\bar{s}]$, $c^*[\bar{s}]$ the norm $\|x\| = \max_{r \in \bar{s}} |x(r)|$. We denote by $v_i^j(f, g)$, $w_i^j(g, f)$ the functions $v(r, z)$, $w(r, z)$, harmonic in the domain $r \geqslant 0$, $z > 0$, which are decreasing at infinity and satisfy the boundary conditions

$$v|_{z=0} = f(r),\ r \in \Omega_i^j,\ \partial v/\partial z|_{z=0} = g(r),\ r \in D\bar{\Omega}_i^j$$
$$\partial w/\partial z|_{z=0} = g(r),\ r \in \Sigma_i^j,\ w|_{z=0} = f(r),\ r \in D\bar{\Sigma}_i^j$$

Here and below, Ds is the complement of the set s up to the semi-axis, i.e., $Ds = [0, \infty) \setminus s$.

Notice that, since $\Omega_1^n = D\bar{\Sigma}_0^n$, $\Sigma_0^n = D\bar{\Omega}_1^n$, we have

$$u(r, z) = v_1^n(f, g) = w_0^n(g, f) \tag{1.3}$$

Let us emphasize that, in the notation $v_i^j(f, g)$, $w_i^j(g, f)$, the f and g are functional arguments. Strictly, the arguments of these functions are r and z.

Problem (1.1), (1.2) has been studied in detail in the case when there are just two lines separating the boundary conditions /4, 5/. We can therefore solve the general case if we can find a method for reducing it to the case of just two lines when there are different relations between the parameters. Our approach below is based on two methods of reducing the problem to two other problems with fewer lines of separation of the boundary conditions.

2. First method. We shall seek the solution $v_1^n(f, g)$ of our problem as

$$v_1^n(f, g) = v_1^m(\alpha, \mu) + v_{m+1}^n(\beta, l) \tag{2.1}$$

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where the functions α and β are given by the system

$$\alpha(r) = f(r) - v_{m+1}^{n}(\beta, l)|_{z=0}, \ r \in \bar{\Omega}_1^m$$
$$\beta(r) = f(r) - v_1^m(\alpha, \mu)|_{z=0}, \ r \in \bar{\Omega}_{m+1}^n \tag{2.2}$$

and functions l, μ are chosen so that

$$\mu(r) + l(r) = g(r), \ r \in \Sigma_0^n \tag{2.3}$$

Obviously, if α, β, μ, l satisfy conditions (2.2), (2.3), Eq.(2.1) becomes an identity.

Thus, on solving system (2.2), we arrive at two independent problems (the determination of $v_1^m(\alpha, \mu)$, $v_{m+1}^n(\beta, l)$, given the functions α, μ, β, l) with fewer lines separating the boundary conditions.

Theorem 1. If $\mu \in c^*[D\Omega_1^m]$, $l \in c^*[D\Omega_{m+1}^n]$, then system (2.2) has a unique solution in the class of continuous functions, which can be obtained by the method of successive approximations with a convergence coefficient not exceeding b_m/a_{m+1}.

Proof. Obviously, if $\alpha \in c[\bar{\Omega}_1^m]$, $\mu \in c^*[D\Omega_1^m]$, $\beta \in c[\bar{\Omega}_{m+1}^n]$, $l \in c^*[D\Omega_{m+1}^n]$, then $v_1^m(\alpha, \mu)|_{z=0} \in c[\bar{\Omega}_{m+1}^n]$ $v_{m+1}^n(\beta, l)|_{z=0} \in c[\bar{\Omega}_1^m]$. Consequently, linear operators A, M, B, L, exist which respectively map $c[\bar{\Omega}_1^m]$ into $c[\bar{\Omega}_{m+1}^n]$, $c^*[D\Omega_1^m]$ into $c[\bar{\Omega}_{m+1}^n]$, $c[\bar{\Omega}_{m+1}^n]$ into $c[\bar{\Omega}_1^m]$, $c[D\Omega_{m+1}^n]$ into $c[\bar{\Omega}_1^m]$, such that

$$v_1^m(\alpha, \mu)|_{z=0} = A\alpha + M\mu, \ v_{m+1}^n(\beta, l)|_{z=0} = B\beta + Ll \tag{2.4}$$

Using the last relations, we can rewrite system (2.2) as

$$\alpha = F_1 - B\beta, \ \beta = F_2 - A\alpha; \ F_1 = f - Ll, \ F_2 = f - M\mu \tag{2.5}$$

We eliminate β from the first of Eqs.(2.5) by using the second equation. We find

$$\alpha = F + BA\alpha, \ F = F_1 - BF_2 \tag{2.6}$$

We estimate the norm of the operator $T = BA$. For this, we note that, if the function $v^m(1, 0)$ is harmonic in the domain $r \geqslant 0$, $z > 0$, and satisfies the conditions

$$v^m(1,0)|_{z=0} = 1, \quad r < b_m, \quad \frac{\partial}{\partial z} v^m(1,0)|_{z=0} = 0, \quad r > b_m$$

then, by the maximum principle for harmonic functions we have (χ is a monotonically decreasing function)

$$v_1^m(1,0)|_{z=0} \leqslant v^m(1,0)|_{z=0}, \quad r \in \bar{\Omega}_{m+1}^n$$
$$v_{m+1}^n(\chi, 0)|_{z=0} \leqslant \chi(a_{m+1}), \quad r \in \bar{\Omega}_1^m; \quad \chi \in c[\bar{\Omega}_{m+1}^n]$$

Taking into account that /3/

$$v^m(1,0)|_{z=0} = \chi^*(r) = \frac{2}{\pi} \arcsin\frac{b_m}{r}, \quad r > b_m$$

we find

$$Ae = v_1^m(1,0)|_{z=0} \leqslant v^m(1,0)|_{z=0} = \chi^*, \ B\chi^* \leqslant \chi^*(a_{m+1}) \tag{2.7}$$

where e is the element of $c[\bar{\Omega}_1^m]$ which takes unity values for all values of the argument. Notice that, given any $x \geqslant 0$ ($x \in c[\Omega_1^m]$) $Tx \geqslant 0$. Hence $\|T\| = \max_{\bar{\Omega}_1^m} Te$. For the proof, it is sufficient to note that, if $\|x\| < 1$, then

$$|Tx| - Te = \begin{cases} T(x-e) \leqslant 0, & Tx \geqslant 0 \\ -T(x+e) \leqslant 0, & Tx < 0 \end{cases}$$

Using inequalities (2.7), we obtain

$$\|T\| = \max_{\bar{\Omega}_1^m} BAe \leqslant \max_{\bar{\Omega}_1^m} B\chi^* \leqslant \chi^*(a_{m+1}) \leqslant b_m/a_{m+1}$$

We now use Banach's fixed point theorem. The theorem is proved.

3. Second method. Along with (2.1), we can write the required harmonic function $v_1^n(f, g) = w_0^n(g, f)$ in the form

$$w_0^n(g, f) = w_0^{m-1}(\eta, p) + w_m^n(\zeta, q) \tag{3.1}$$

if the functions η, p, ζ, q satisfy the conditions

$$\eta(r) = G_1(r) - \frac{\partial}{\partial z} w_m^n(\zeta, 0)|_{z=0}, \quad r \in \bar{\Sigma}_0^{m-1} \tag{3.2}$$

$$\zeta(r) = G_2(r) - \frac{\partial}{\partial z} w_0^{m-1}(\eta, 0)|_{z=0}, \quad r \in \bar{\Sigma}_m{}^n$$
$$G_1(r) = g(r) - \frac{\partial}{\partial z} w_m{}^n(0, q)|_{z=0}, \quad G_2(r) = g(r) - \frac{\partial}{\partial z} w_0^{m-1}(0, p)|_{z=0}$$
$$p(r) + q(r) = f(r), \; r \in D\Sigma_0{}^n \tag{3.3}$$

We can regard Eq.(3.3) as a constraint on the choice of the functions p, q, while (3.2) is a system of equations for finding the functions η, ζ.

Theorem 2. If $G_1 \in c\,[\bar{\Sigma}_0{}^{m-1}]$, $G_2 \in c^*\,[\bar{\Sigma}_m{}^n]$, then system (3.2) has a unqiue solution in the class of continuous functions, which can ge obtained by the method of successive approximations with convergence coefficient not exceeding $(a_m/b_m)^3$.

Proof. Obviously, if $\eta \in c\,[\bar{\Sigma}_0{}^{m-1}]$, $\zeta \in c^*\,[\bar{\Sigma}_m{}^n]$, then

$$\frac{\partial}{\partial z} w_0^{m-1}(\eta, 0)|_{z=0} \in c^*[\bar{\Sigma}_m^n], \quad \frac{\partial}{\partial z} w_m{}^n(\zeta, 0)|_{z=0} \in c[\bar{\Sigma}_0^{m-1}]$$

Consequently, there are linear operators Z and Y which respectively map $c^*\,[\bar{\Sigma}_m{}^n]$ into $c\,[\bar{\Sigma}_0{}^{m-1}]$ and $c[\bar{\Sigma}_0{}^{m-1}]$ into $c^*\,[\bar{\Sigma}_m{}^n]$, such that

$$\frac{\partial}{\partial z} w_0^{m-1}(\eta, 0)|_{z=0} = Y\eta, \quad \frac{\partial}{\partial z} w_m{}^n(\zeta, 0)|_{z=0} = Z\zeta$$

On eliminating the function ζ from the first of Eqs.(3.2) by means of the second, and introducing the notation

$$\xi = \tau\eta, \; R = \tau(G_1 + YG_2), \; X\xi = \tau ZY(y\xi)$$
$$\tau(r) = b_m{}^2 - r^2, \; y(r) = (b_m{}^2 - r^2)^{-1}$$

we obtain the equation $\xi = R + X\xi$. Since τ, $y \in c\,[\bar{\Sigma}_0{}^{m-1}]$, to prove the theorem it suffices to show that $\| X \| \leqslant (a_m/b_m)^3$.

We note that, if $x_1 \in c\,[0, a_m]$, $x_2 \in c^*\,[\omega]$, $\omega = [b_m, \infty)$, $x_1 \geqslant 0$, $x_2 \geqslant 0$, then

$$0 \leqslant -\frac{\partial}{\partial z} w^{m-1}(x_1, 0)|_{z=0} \leqslant -\frac{\partial}{\partial z} w^{m-1}(x_1, 0)|_{z=0}, \quad r \in \bar{\Sigma}_m{}^n \tag{3.4}$$
$$0 \leqslant -\frac{\partial}{\partial z} w_m{}^n(x_2, 0)|_{z=0} \leqslant -\frac{\partial}{\partial z} w_m(x_2, 0)|_{z=0}, \quad r \in \bar{\Sigma}_0^{m-1} \tag{3.5}$$

where $w^{m-1}(x_1, 0)$, $w_m(x_2, 0)$ are harmonic in the domain $r > 0$, $z > 0$, and satisfy the conditions

$$\frac{\partial}{\partial z} w^{m-1}(x_1, 0)|_{z=0} = x_1(r), r \leqslant a_m, \quad w^{m-1}(x_1, 0)|_{z=0}, \quad r > a_m$$
$$\frac{\partial}{\partial z} w_m(x_2, 0)|_{z=0} = x_2(r), \quad r > b_m, \; w_m(x_2, 0)|_{z=0} = 0, \quad r < b_m$$

The left-hand sides of (3.4), (3.5) may be proved in the same way as in /6, p.223/. To prove the right-hand side of (3.4), we only need to observe that the functions $s = w_0{}^{m-1}(x_1, 0) - w^{m-1}(x_1, 0)$ satisfy the conditions

$$\partial s/\partial z|_{z=0} \leqslant 0, \; r \in (0, a_m), \; s|_{z=0} = 0, \; r \in (a_m, \infty)$$

The right-hand side of (3.5) is proved in a similar way. Noting that /3/

$$\frac{\partial}{\partial z} w^{m-1}(x_1, 0)|_{z=0} = -\frac{2}{\pi}\frac{1}{\sqrt{r^2 - a_m^2}} \int_0^{a_m} \frac{\rho\sqrt{a_m^2 - \rho^2}\, x_1(\rho)}{r^2 - \rho^2} d\rho, \quad r \in \bar{\Sigma}_m{}^n$$
$$\frac{\partial}{\partial z} w_m(x_2, 0)|_{z=0} = -\frac{2}{\pi}\frac{1}{\sqrt{b_m^2 - r^2}} \int_{b_m}^{\infty} \frac{\rho\sqrt{\rho^2 - b_m^2}\, x_2(\rho)}{\rho^2 - r^2} d\rho, \quad r \in \bar{\Sigma}_0^{m-1}$$

we have

$$\| X \| \leqslant \max_{r \in \varkappa} \frac{4}{\pi^2} \sqrt{b^2 - r^2} \int_b^{\infty} \frac{t\sqrt{t^2 - b^2}\,dt}{(t^2 - r^2)\sqrt{t^2 - a^2}} \int_0^a \frac{\rho\sqrt{a^2 - \rho^2}\,d\rho}{(t^2 - \rho^2)(b^2 - \rho^2)} \leqslant$$
$$\frac{a^3}{b^3} \max_{r \in \varkappa} \frac{2}{\pi} \sqrt{b^2 - r^2} \int_b^{\infty} \frac{t\sqrt{t^2 - b^2}\,dt}{(t^2 - r^2)(t^2 - a^2)} \leqslant \frac{a^3}{b^3}$$
$$\varkappa = \bar{\Sigma}_0{}^{m-1}, \; a = a_m, \; b = b_m$$

The theorem is proved.

Notice that the conditions of the theorem are satisfied if the functions p and q are chosen e.g., in the following way:

$$p \in c\,[D\Sigma_0^{m-1}],\ q \in c^*\,[D\Sigma_m^{\ n}];\ p(r) = 0,\ r > r_2$$
$$q(r) = 0,\ r < r_1;\ r_1, r_2 \in (a_m, b_m),\ r_1 < r_2$$

4. Construction of an effective solution. The above methods enable us to reduce a problem with $2n$ lines dividing the boundary conditions to two problems with $2m$ and $2(n-m)$ dividing lines respectively. One of the above methods can again be applied to each of the resulting problems. By continuing this process, we can in the long run arrive at a problem with two lines dividing the boundary conditions. Realization of the process requires the solution of system (2.2) or (3.2), depending on the method chosen at a given step. The convergence coefficient of the iterations when solving system (2.2) by the method of successive approximations does not exceed b_m/a_{m+1}, or when solving system (3.2), $(a_m/b_m)^3$. In view of this, the following algorithm can be stated for choosing the method of solution: if $b_i/a_{i+1} \leqslant (a_j/b_j)^3$, we must use the first method, putting $m = i$, and otherwise, the second method, putting $m = j$. Here, i denotes the value of k at which $\min(b_k/a_{k+1})$ is reached, and j the value of k at which $\min(a_k/b_k)$ is reached. In both cases, k runs over all possible values except for those at which the ratio b_k/a_{k+1} or a_k/b_k is zero.

As an example, consider the axisymmetric problem on the joint indentation of ring and circular dies into an elastic half-space. We assume that the dies are rigid, have plane bases, and indent without friction. We also assume that the surface of the half-space external to the dies is free from stresses.

Using the Pankovich-Neiber relations, we reduce the problem to finding the harmonic function $u(r, z)$ which satisfies the boundary conditions

$$u|_{z=0} = G(1-\nu)^{-1}\varepsilon_k,\quad r \in (a_k, b_k),\quad k = 1, 2,\quad a_1 = 0 \tag{4.1}$$
$$\partial u/\partial z|_{z=0} = 0,\quad r \in (b_1, a_2) \cup (b_2, \infty)$$

where ε_k are the die displacements, G is the shift modulus, ν is Poisson's ratio. In accordance with our above notation (δ_{jk} is the Kronecker delta)

$$u = \frac{G}{1-\nu}\sum_{j=1}^{2}\varepsilon_j v_1^{\,2}(f_j, 0);\quad f_j(r) = \delta_{jk},\quad a_k < r < b_k \tag{4.2}$$

Let the distance between the dies be large (the ratio b_1/a_2 is small). Using the first method of reduction to two problems with fewer lines dividing the boundary conditions, we write the required functions in the form $v_1^{\,2}(f_j, 0) = v_1^{\,1}(f_{j1}, 0) + v_2^{\,2}(f_{j2}, 0)$, where the functions f_{jk} are given by system (2.2) with $m = 1,\ n = 2,\ \alpha = f_{j1},\ \beta = f_{j2},\ f = f_j,\ l = 0,\ \mu = 0$. This system can be solved by successive approximations, with a convergence coefficient not exceeding b_1/a_2. Thus our method works well in the present case.

The functions $v_1^{\,1}(f_{j1}, 0),\ v_2^{\,2}(f_{j2}, 0)$ are the solutions of problems with one and two lines dividing the boundary conditions. On writing these solutions in the form /7/

$$v_k^{\,k}(f_{jk}, 0) = \int_0^\infty V_{jk}(\lambda)\, J_0(\lambda r)\, e^{-\lambda z}\, d\lambda \tag{4.3}$$
$$V_{j1} = \int_0^a t_1\psi_{j1}(t)\sin\lambda t\, dt + \int_a^\infty t_1\omega_{j1}(t)\cos\lambda t\, dt$$
$$V_{j2} = \int_0^b t_2\varphi_{j2}(t)\sin\lambda t\, dt + \int_b^c t_2\psi_{j2}(t)\cos\lambda t\, dt + \int_c^\infty t_2\omega_{j2}(t)\cos\lambda t\, dt$$
$$t_1 = t/\sqrt{|a^2 - t^2|},\quad t_2 = t/\sqrt{|c^2 - t^2|},\quad a = b_1,\quad b = a_2,\quad c \doteq b$$

and rewriting system (2.2) with respect to the functions $\psi_{jk},\ \omega_{jk},\ \varphi_{j2}$, we obtain

$$\psi_{j1}(t) = \frac{2}{\pi}\delta_{j1} - \frac{t_2}{t_1}\varphi_{j2}(t) - \int_a^b \tau\vartheta\tau_2\varphi_{j2}(\tau)\frac{d\tau}{\tau_1},\quad t \leqslant a \tag{4.4}$$
$$\omega_{j1}(t) = t_1^{-1}\int_0^a \tau\tau_1\vartheta\psi_{j1}(\tau)\, d\tau,\quad t \geqslant a$$
$$\varphi_{j2}(t) = \frac{t_2}{t}\left[\int_b^c \tau\vartheta\psi_{j2}(\tau)\, d\tau + \int_c^\infty \tau_2\vartheta\omega_{j2}(\tau)\, d\tau\right],\quad t \leqslant b$$

$$\psi_{j2}(t)=\frac{2}{\pi}\delta_{j2}-\frac{t_1}{t_2}\omega_{j1}(t)-\int_a^b \tau\tau_1\vartheta\omega_{j1}(\tau)\frac{d\tau}{\tau_2},\quad b\leqslant t\leqslant c$$

$$\omega_{j2}(t)=t_2\int_0^b \tau\tau_2\vartheta\varphi_{j2}(\tau)\,d\tau,\quad t\geqslant c$$

$$\vartheta=2\pi^{-1}/\mid t^2-\tau^2\mid,\quad \tau_1=\tau/\sqrt{\mid a^2-\tau^2\mid},\quad \tau_2=\tau/\sqrt{\mid b^2-\tau^2\mid}$$

On solving system (4.4), we obtain with the aid of (4.2) and (4.3) the solution of our problem.

It can be shown that system (4.4) has a unique solution in the class of continuous functions, which can be obtained by successive approximations, with convergence coefficient not exceeding $\max[b_1/a_2, (a_2/b_2)^3]$. The presence of the coefficient $(a_2/b_2)^3$ is linked to the fact that the problem of finding $v_2^2(f_{j2}, 0)$ reduces to a Fredholm integral equation of the 2nd kind, in the solution of which by successive approximation the convergence coefficient of the iterations does not exceed $(a_2/b_2)^3$ *. (*BELYAYEV S.YU., The mixed problem on the deformation of and elastic half-space with any number of circular concentric lines separating the boundary conditions, Candidate Dissertion, LIP im. M.I. Kalinin, Leningrad, 1983.).

The most important characteristics of the problem are expressible in terms of the solution of system (4.4). For example, consider the numerical connection of displacements ε_k with the forces P_k acting on the dies. This connection is given by the cofficients p_{kj}:

$$P_k=Gc(1-\nu)^{-1}\sum_{j=1}^{2}p_{kj}\varepsilon_j \tag{4.5}$$

$$p_{1j}=\frac{2\pi}{c}\int_a^\infty t_1\omega_{j1}(t)\,dt \tag{4.6}$$

$$p_{2j}=\frac{2\pi}{c}\left[\int_b^c t_2\psi_{j2}(t)\,dt+\int_c^\infty t_2\omega_{j2}(t)\,dt\right]$$

Now let the distance between the dies be small $(b_1/a_2\approx 1)$. Using the second method of reduction to two problems with fewer lines separating the boundary conditions, we obtain

$$u(r,z)=\int_0^\infty [W_{j1}(\lambda)+W_{j2}(\lambda)]\,J_0(\lambda r)\,e^{-\lambda z}\,d\lambda$$

$$W_{j1}=\int_0^a t_1\alpha_{j1}(t)\cos\lambda t\frac{dt}{t}+\int_a^b t_1\beta_{j1}(t)\sin\lambda t\frac{dt}{t}+\int_b^\infty t_1\gamma_{j1}(t)\cos\lambda t\frac{dt}{t}$$

$$W_{j2}=\int_0^c \beta_{j2}(t)\cos\lambda t\,dt+\int_c^\infty \gamma_{j2}(t)\cos\lambda t\,dt$$

where the functions α_{ji}, β_{jk}, γ_{jk} are found from the system

$$\alpha_{j1}(t)=t_2\left[\delta_{j1}-\delta_{j2}-\int_a^b \tau_2\vartheta\beta_{j1}(\tau)\,d\tau\right],\quad t\leqslant a \tag{4.7}$$

$$\beta_{j2}(t)=\frac{2}{\pi}\Delta+t_1t_2\left[\int_b^\infty \tau_1\tau_2\vartheta\gamma_{j1}(\tau)\frac{d\tau}{\tau}+\int_c^\infty \tau_1\vartheta\gamma_{j2}(\tau)\frac{d\tau}{\tau}\right],\quad a\leqslant t\leqslant b$$

$$\gamma_{j1}(t)=-t\int_0^a \vartheta\alpha_{j1}(\tau)\,d\tau,\quad t\geqslant b,\quad \beta_{j2}(t)=\frac{2}{\pi}\delta_{j2},\quad t\leqslant c$$

$$\gamma_{j2}(t)=-t^{-1}t_2\gamma_{j1}(t)+\int_a^b \tau_2\vartheta\beta_{j1}(\tau)\,d\tau,\quad t\geqslant c$$

$$\Delta=2\pi^{-1}\delta_{j2}t^{-1}t_2\ln\left[\left(\sqrt{c^2-a^2}+\sqrt{t^2-a^2}\right)/\sqrt{c^2-t^2}\right]$$

This system has a unique solution in the class of continuous functions, which can be found by successive approximations with convergence coefficient not exceeding $\max[(a_2/b_2)^3, {}^1/_2]$. The coefficients p_{kj} in (4.5) are expressible in terms of the solution of system (4.7) by

$$p_{1j}=\frac{2\pi}{c}\left[\frac{2}{\pi}\delta_{j2}\left(c-\sqrt{c^2-a^2}\right)+\int_0^a t_2\alpha_{j2}(t)\frac{dt}{t}+\right. \tag{4.8}$$

$$\left.\int_b^\infty (1-t_1)\,t_2\gamma_{j1}(t)\frac{dt}{t}+\int_c^\infty (1-t_1)\,\gamma_{j2}(t)\,dt\right]$$

$$p_{2j}=\frac{2\pi}{c}\left[\frac{2}{\pi}\delta_{j2}\left(\sqrt{c^2-a^2}-\sqrt{b^2-a^2}\right)+\int_a^b t_1t_2\beta_{j1}(t)\frac{dt}{t}+\int_b^c t_1t_2\gamma_{j1}(t)\frac{dt}{t}\right]$$

The results of computations from (4.6), (4.8) with $b_2 = 1$, $a_2 = 0,5$ and different $x = b_1/a_2$ are shown in Fig.1, where

$$q_{kj} = c_{kj}p_{kj}/(1 - \ln(1 - b_1/a_2)),\quad c_{11} = 1,\quad c_{12} = -1,\quad c_{22} = 1/3 \qquad (4.9)$$

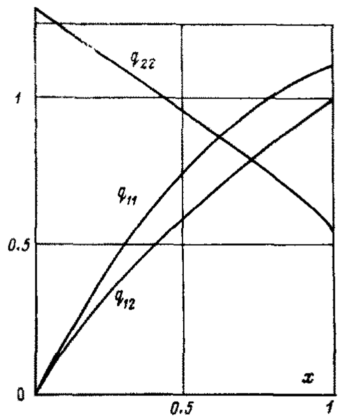

Fig.1

The coefficients q_{kj} are introduced so as to isolate the logarithmic singularity of p_{kj} in the neighbourhood of the point $b_1/a_2 = 1$. However, it is the coefficients p_{kj} that have a physical meaning. Using Fig.1 and relations (4.9), we can show that p_{11} increases as the distance between the dies decreases (and has a logarithmic singularity at $b_1/a_2 = 1$). This behaviour of p_{11} as a function of b_1/a_2 is explained by the fact that, apart from a constant factor, it is equal to the force P_1 that has to be applied to the inner die in order for it to sink to unit depth in the half-space when the normal displacements under the outer die are kept equal to zero.

The coefficient p_{22} has a similar meaning. The difference in the behaviour of p_{11} and p_{22} is that, as $b_1/a_2 \to 0$, we have $p_{11} \to 0$, while $p_{22} \to \text{const} > 0$. The difference is due to the decrease in the base area of the inner die to zero as $b_1/a_2 \to 0$.

As regards the coefficient $p_{12} = p_{21}$, it characterizes the mutual influence of the dies on one another and behaves qualitatively in the same way as p_{11}, but takes negative values. This is because, gives the same force P_1, the displacement ε_1 is greater if $\varepsilon_2 > 0$.

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